Chapter 3 Arithmetic Functions and Dirichlet Series.

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[**4 lectures**]

3.1 Arithmetic Functions

Definition 3.1 *An **arithmetic function** is any function $f : \mathbb{N} \to \mathbb{C}$.*

Examples

1) *The **divisor function** $d(n)$ (often denoted $\tau(n)$) is the number of divisors of n, i.e.*

$$d(n) = \sum_{d|n} 1.$$

Note that if $d|n$ then $n = d(n/d) = ab$ say, so $d(n)$ equals the number of pairs of integers (a, b) with product n, i.e.

$$d(n) = \sum_{ab=n} 1.$$

This can be extended to give $d_k(n)$, the number of ordered k-tuples of integers $(a_1, a_2, ..., a_k)$ with product n, i.e.

$$d_k(n) = \sum_{a_1 a_2 \ldots a_k = n} 1.$$

2) *The function $\sigma(n)$ is the sum of divisors of n, i.e.*

$$\sigma(n) = \sum_{d|n} d.$$

This can be extended, so for any $\nu \in \mathbb{C}$ define

$$\sigma_\nu(n) = \sum_{d|n} d^\nu.$$

3) *The **Euler Totient** or **phi function** is the number of integers $\leq n$ which are co-prime to n, so*

$$\phi(n) = \sum_{\substack{1 \leq r \leq n \\ \gcd(r,n)=1}} 1.$$

Students may have seen $\phi(n)$ as the number of invertible elements in $\mathbb{Z}_n$, or equivalently the cardinality of $\mathbb{Z}_n^*$.

4) *The function* $\omega(n)$ *counts the number of* ***distinct*** *prime factors of* n, *so*

$$\omega(n) = \sum_{p|n} 1.$$

Recall the notation $\|$ defined by $p^a \| n$ if, and only if, a is the largest power of p that divides n. So $p^a \| n$ if, and only if, $p^a | n$ and $p^{a+1} \nmid n$.

5) *Then define* $\Omega(n)$ *to be the number of primes that divide* n ***counted with multiplicity***, *i.e.*

$$\Omega(n) = \sum_{p^a \| n} a = \sum_{p} \sum_{\substack{r \geq 1 \\ p^r | n}} 1.$$

Some further important examples of arithmetic functions are

$$\begin{aligned}
\bullet)\ 1(n) &= 1 \text{ for all } n, \\
\bullet)\ j(n) &= n \text{ for all } n, \text{ sometimes known as } id, \\
\bullet)\ \delta(n) &= \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{sometimes known as } e(n).
\end{aligned}$$

To an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ we can associate a generating function.

Definition 3.2 *Given* $f : \mathbb{N} \to \mathbb{C}$ *define the* ***Dirichlet Series of*** f *to be*

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

defined for those $s \in \mathbb{C}$ *at which the series converges.*

In this Chapter we are not particularly interested in the question of where a given Dirichlet Series converges, though the answer is known in the following example:

Example 3.3 *If* $f = 1$ *then*

$$D_1(s) = \sum_{n=1}^{\infty} \frac{1(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s),$$

the Riemann zeta function, which is known to converge when $\operatorname{Re} s > 1$ *and converges absolutely only in this half plane.*

If $f = j$ *then*

$$D_j(s) = \sum_{n=1}^{\infty} \frac{j(n)}{n^s} = \sum_{n=1}^{\infty} \frac{j}{n^s} = \zeta(s-1)\,,$$

valid for $\operatorname{Re}(s-1) > 1$*, i.e.* $\operatorname{Re} s > 2$.
If $f = \delta$ *then*

$$D_\delta(s) = \sum_{n=1}^{\infty} \frac{\delta(n)}{n^s} = 1,$$

valid for all $s \in \mathbb{C}$.

3.2 Convolution and Dirichlet Series.

Recall (or see Background Notes 0.2) that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{m=1}^{\infty} b_m$ are absolutely convergent with sums A and B respectively then the double sum $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n b_m$ converges absolutely to AB. Being absolutely convergent means we can rearrange the double summation however we like. In particular we can group together $a_n b_m$ in different ways and then sum over the groups. For example

Definition 3.4 *The* ***Cauchy Product*** *(or* ***Convolution****) of* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{m=0}^{\infty} b_m$ *is to collect all terms* $a_n b_m$ *for which the* sum $n+m$ *is constant, i.e.* $n + m = N$*. That is*

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{m=0}^{\infty} b_m\right) = \sum_{N=0}^{\infty}\left(\sum_{m+n=N} a_n b_m\right).$$

This has applications to the products of power series. Alternatively (and note now that the sums start at 1 and not 0 as before),

Definition 3.5 *The* ***Dirichlet Product*** *(or* ***Convolution****) of* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{m=1}^{\infty} b_m$ *is to collect all terms* $a_n b_m$ *for which the* product nm *is constant, i.e.* $nm = N$*. That is*

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{m=1}^{\infty} b_m\right) = \sum_{N=1}^{\infty}\left(\sum_{mn=N} a_n b_m\right).$$

Let f and g be two arithmetic functions such that their associated Dirichlet Series converge absolutely at some $s \in \mathbb{C}$. Then, at this s we can look at the Dirichlet convolution of the two series,

$$\begin{aligned} D_f(s)\, D_g(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{m=1}^{\infty} \frac{g(m)}{m^s} = \sum_{N=1}^{\infty} \left(\sum_{nm=N} \frac{f(n)}{n^s} \frac{g(m)}{m^s} \right) & (1) \\ &= \sum_{N=1}^{\infty} \frac{1}{N^s} \left(\sum_{nm=N} f(n)\, g(m) \right). & (2) \end{aligned}$$

So the Dirichlet product of Dirichlet series of arithmetic functions gives a Dirichlet Series of a new arithmetic function. This motivates the

Definition 3.6 *If f and g are arithmetic functions then the **Dirichlet Convolution** $f * g$ is defined by*

$$f * g(n) = \sum_{ab=n} f(a)\, g(b) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) \qquad (3)$$

for all $n \in \mathbb{N}$.

Algebraic Properties

Commutativity For any $n \geq 1$ we have

$$\begin{aligned} f * g(n) &= \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) \text{ from third term in } (3) \\ &= \sum_{d|n} g\left(\frac{n}{d}\right) f(d) \\ &\quad \text{since multiplication of real numbers is commutative} \\ &= g * f(n) \text{ from the fourth term in } (3)\,. \end{aligned}$$

Then $f * g(n) = g * f(n)$ true for all n means $f * g = g * f$, i.e. Dirichlet Convolution, $*$, is commutative.

Associativity Given three arithmetic functions f, g and h, note that

$$\begin{aligned}(f * g) * h &= \sum_{ab=n} (f * g)(a)\, h(b) = \sum_{ab=n} \left(\sum_{cd=a} f(c)\, g(d) \right) h(b) \\ &= \sum_{cdb=n} f(c)\, g(d)\, h(b) \\ &= \sum_{a_1 a_2 a_3 = n} f(a_1)\, g(a_2)\, h(a_3)\,, \end{aligned} \tag{4}$$

on relabeling. The symmetry of the final sum shows that

$$(f * g) * h = f * (g * h)\,,$$

i.e. $*$ is associative, and we can simply write $f * g * h$ with no ambiguity.

Identity The following has been left to the student

Exercise 3.7 *on Problem Sheet. Recall $\delta(n) = 1$ if $n = 1$, 0 otherwise. Show that*

$$f * \delta = \delta * f = f$$

for all arithmetic functions f.

Thus δ is the **identity** under $*$.

Inverses As soon as you have an identity you ask if there are inverses. That is, for what f does there exist g such that $f * g = \delta$? In the appendix it is shown that f has an inverse under $*$ if, and only if, $f(1) \neq 0$.

Distributivity It is left as an example on the problem sheets to show that for arithmetic function f, g and h we have

$$f * (g + h) = f * g + f * h.$$

Examples of Convolution

Example 3.8 *For $n \geq 1$ we have*

$$1 * 1(n) = \sum_{d|n} 1(d)\, 1\left(\frac{n}{d}\right) = \sum_{d|n} 1 = d(n)\,,$$

thus $d = 1 * 1$.

Starting from the definition of d_k as the number of ordered k-tuples whose product of entries is n we find

$$\begin{aligned} d_k(n) &= \sum_{a_1 a_2 \ldots a_k = n} 1 = \sum_{a_1 a_2 \ldots a_k = n} 1(a_1)\, 1(a_2) \ldots 1(a_k) \\ &= \underbrace{1 * 1 * \ldots . * 1}_{k \text{ times}}(n)\,, \end{aligned}$$

by a generalization of (4) . Thus

$$d_k = \underbrace{1 * 1 * \ldots . * 1}_{k \text{ times}}.$$

Example 3.9 *For $n \geq 1$ we have*

$$1 * j(n) = \sum_{d|n} 1\left(\frac{n}{d}\right) j(d) = \sum_{d|n} d = \sigma(n)\,,$$

so $\sigma = 1 * j$. Similarly $\sigma_\nu = 1 * j^\nu$.

In these examples we take products of arithmetic functions and see what we obtain. Can we instead take an arithmetic function, such as Euler's ϕ-function, and see how it factorises?

Dirichlet Series

The motivation for the introduction of the Dirichlet convolution, (2), can be written as

$$D_f(s)\, D_g(s) = D_{f*g}(s)\,,$$

for all s for which the Dirichlet Series on the left converge absolutely. This leads to some examples

Example 3.10

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(n)}{n^s} &= D_d(s) = D_{1*1}(s) \\ &= D_1(s)\, D_1(s) \qquad \textit{by } (2) \\ &= \zeta^2(s)\,. \end{aligned}$$

Since the individual factors $\zeta(s)$ converge absolutely for $\operatorname{Re} s > 1$*, the final result holds for* $\operatorname{Re} s > 1$.

Example 3.11 *Similarly*

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta^k(s)$$

for $\mathrm{Re}\, s > 1$.

Example 3.12 *The Dirichlet series for the sum of divisors function* σ *is*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} &= D_\sigma(s) = D_{1*j}(s) \\ &= D_1(s)\, D_j(s) \qquad \text{by } (2) \\ &= \zeta(s)\, \zeta(s-1)\,,\end{aligned}$$

using Example 3.3, valid for $\mathrm{Re}\, s > 1$ *and* $\mathrm{Re}\,(s-1) > 1$, *i.e.* $\mathrm{Re}\, s > 2$.

3.3 Multiplicative and Additive Functions

Definition 3.13 *An arithmetic function is* ***multiplicative*** *if,*

$$\forall m, n \in \mathbb{N}, \ \ \gcd(m,n) = 1 \Rightarrow f(mn) = f(m)\, f(n)\,,$$

completely *(or* ***totally****)* ***multiplicative*** *if*

$$\forall m, n \in \mathbb{N}, \ f(mn) = f(m)\, f(n)\,,$$

strongly multiplicative *if multiplicative and*

$$\forall p, \, \forall r \geq 1, \ f(p^r) = f(p)\,.$$

Note Factoring $n = \prod_{p^a \| n} p^a$ into distinct primes then f multiplicative means that

$$f(n) = \prod_{p^a \| n} f(p^a)\,. \tag{5}$$

So we need only evaluate f *on prime powers.* If f is completely multiplicative then

$$f(n) = \prod_{p^a \| n} f(p)^a\,,$$

So we need only evaluate f *on primes.* If f is strongly multiplicative then

$$f(n) = \prod_{p^a \| n} f(p^a) = \prod_{p^a \| n} f(p) = \prod_{p | n} f(p)\,.$$

Example 3.14 *i.*

$$1(mn) = 1 = 1 \times 1 = 1(m)\,1(n)$$

for all m, n. *Also*

$$1(p^r) = 1 = 1^r = 1(p)^r\,,$$

and so 1 *is completely and strongly multiplicative.*

ii

$$j(mn) = mn = j(m)\,j(n)$$

for all m, n. *Also*

$$j(p^r) = p^r = j(p)^r\,,$$

and so j *is completely and strongly multiplicative.*

iii. If $mn = 1$ *then both* $m = n = 1$, *thus* $\delta(mn) = 1$ *and both* $\delta(m) = \delta(n) = 1$. *Hence* $\delta(mn) = 1 = \delta(m)\,\delta(n)$.

If $mn > 1$ *then at least one of* m *or* n *is* > 1. *Thus* $\delta(mn) = 0$ *and at least one of* $\delta(m)$ *or* $\delta(n)$ *equals 0. Hence* $\delta(mn) = 0 = \delta(m)\,\delta(n)$.

Finally $\delta(p^r) = 0 = \delta(p)^r$.

There was no requirement in that discussion for $\gcd(m, n) = 1$ *therefore* δ *is completely and strongly multiplicative.*

For later use we observe that

Lemma 3.15 *If* f *is a non-zero multiplicative function then* $f(1) = 1$.

Proof If f is non-zero then there exists n for which $f(n) \neq 0$. Yet multiplicatively means

$$f(n) = f(1 \times n) = f(1)\,f(n)\,.$$

Divide by the non-zero $f(n)$ to get $1 = f(1)$. ■

An immediate Corollary of the definition is

Corollary 3.16 *If* f_1 *and* f_2 *are two multiplicative functions satisfying* $f_1(p^r) = f_2(p^r)$ *for all prime powers* p^r *then* $f_1 = f_2$.

Proof Being multiplicative implies $f_1(1) = 1 = f_2(1)$. Assume $n > 1$ then

$$\begin{aligned} f_1(n) &= f_1\left(\prod_{p^a \| n} p^a\right) = \prod_{p^a \| n} f_1(p^a) \\ &= \prod_{p^a \| n} f_2(p^a) \quad \text{by given assumption} \\ &= f_2\left(\prod_{p^a \| n} p^a\right) = f_2(n)\,. \end{aligned}$$

True for all $n \geq 1$ implies $f_1 = f_2$. ■

Definition 3.17 *An arithmetic function is* ***additive*** *if,*

$$\forall m, n \in \mathbb{N}, \;\; \gcd(m, n) = 1 \Rightarrow g(mn) = g(m) + g(n)\,,$$

completely *(or* ***totally****)* ***additive*** *if*

$$\forall m, n \in \mathbb{N}, \;\; g(mn) = g(m) + g(n)\,,$$

strongly additive *if additive and*

$$\forall p, \forall r \geq 1, \;\; g(p^r) = g(p)\,.$$

Note Factoring $n = \prod_{p^a \| n} p^a$ into distinct primes then g additive means that

$$g(n) = \sum_{p^a \| n} g(p^a)\,. \tag{6}$$

So we need only evaluate g on prime powers. If f is completely additive then

$$g(n) = \sum_{p^a \| n} a g(p)\,,$$

while if it is strongly additive then

$$g(n) = \sum_{p^a \| n} g(p^a) = \sum_{p^a \| n} g(p) = \sum_{p | n} g(p)\,.$$

Student to check that if g is a non-zero additive function then $g(1) = 0$.

Important So we see from (6) and (5) that to calculate multiplicative and additive functions it suffices to find their values only on prime powers (and only on primes if either strongly multiplicative or strongly additive).

Example 3.18 *i. By definition,*

$$\omega(n) = \sum_{p|n} 1 = \sum_{p|n} \omega(p)\,,$$

since $\omega\,(p) = 1$ *and so* ω *is strongly additive.*

ii. By definition

$$\Omega(n) = \sum_{p^a \| n} a = \sum_{p^a \| n} a\,\Omega(p)\,,$$

since $\Omega\,(p) = 1$ *and so* Ω *is completely additive.*

Note if g is additive then α^g is multiplicative for any $\alpha \in \mathbb{C}$. In particular 2^ω and 2^Ω are multiplicative, as will be used later.